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1 Abstract

This paper presents analytical solutions for a steady turbulent miscible gravity current flowing along a horizontal rigid boundary of finite length into a quiescent uniform environment. These solutions are obtained from the governing equations (mass, momentum and buoyancy) originally proposed by Ellison & Turner (*J. Fluid. Mech.*, vol. 6, 1959, pp. 423-448) for a Boussinesq buoyant layer. We first derived a system of coupled ordinary differential equations describing the longitudinal evolution of the mean velocity u, the height h, the mean density deficit η and the Richardson number Ri of the current. The theoretical approach described here provides, for an initially supercritical flow (Ri < 1), explicit relations to determine u, h and η solely from the longitudinal evolution of the Richardson number Ri. This evolution is theoretically obtained from a universal function F which can be tabulated and, as in the present paper, also plotted. In addition, the function F allows us to determine (and only from the knowledge of the initial conditions) whether the flow remains supercritical over the whole length of the rigid boundary, or might transit towards a subcritical state (Ri > 1). In this latter case, the method is extended by including a hydraulic jump. The location and amplitude of this jump are calculated from an additional universal function G and the injection conditions.

2 Introduction

A gravity current is a canonical flow that occurs when a light (heavy) fluid propagates into a heavier (lighter) ambient fluid along a rigid boundary. This flow can involve immiscible or miscible fluids. In the latter case, the current engulfs the surrounding fluid in a process called entrainment, resulting in a longitudinal evolution of the current mass flow rate.

Gravity currents arise in many environmental flows such as katabatic winds or oceanic deep currents, to name but a few. They may also appear in hazardous situations such as oil spreading on the sea or fire-induced smoke propagation.

Because of both their ubiquity and academic interest, gravity currents have been widely studied. The pioneering works of Von Kármán (1940) and later Benjamin (1968) were the first to tackle this flow theoretically. One of the main objectives of these works was to determine the dynamics of the current head during the propagation phase. Several authors have subsequently addressed the transient evolution of this flow by proposing models for gravity currents resulting from a fixed-volume release (Huppert & Simpson (1980); Rottman & Simpson (1983); Lowe, Rottman & Linden (2005); Birman, Battandier, Meiburg & Linden (2007); Sher & Woods (2015)) as well as from a fixed-flux release (Huppert (1982); Didden & Maxworthy (1982); Shin, Dalziel & Linden (2004); Marino, Thomas & Linden (2005); Ungarish (2017)).

In contrast to the transient phase of gravity currents, that has been addressed in a substantial number of studies, the steady phase (i.e. after a long time subsequent to the reaching of the exit of the rigid boundary by the flow) remains little investigated so far. In their seminal article, Ellison & Turner (1959) (hereafter referred to as ET59) have developed a theoretical model for a fixed-flux steady gravity current based on the conservation equations for mass, momentum and buoyancy. Through a system of coupled differential equations, their model allows the evolution of the three variables of the current (velocity, thickness and density) to be calculated along the longitudinal propagation $x-$axis. Similarly to Morton, Taylor & Turner (1956) for turbulent plumes, they introduced an entrainment coefficient E to quantify the amount of ambient fluid entrained into the current. After some algebraic manipulations, the Richardson number Ri naturally appears in their equations. It is defined as $\mathrm{Ri} = \Delta\rho g h/\rho_a u^2$, with $\Delta\rho = |\rho_a - \rho|$, ρ_a the density of the ambient, g the gravitational acceleration and ρ, h and u, the mean density, thickness and mean velocity of the current, respectively. The Richardson number, which characterises the local stability of the current, allows three different regimes to be identified: the supercritical regime when $\mathrm{Ri} < 1$, the subcritical regime when $\mathrm{Ri} > 1$ and a so-called critical regime when $\mathrm{Ri} = 1$. In addition, it allows the entrainment coefficient to be estimated, as it will be mentioned in the next section.

Although the paper of ET59 has been widely cited, it is surprising that solving

the equations of their model has not received much attention. The difficulty in solving these equations comes from the fact that a mathematical singularity occurs if Ri = 1, i.e. when the flow reaches the critical regime.

In their attempt to solve the ET59 equations, Guo, Li, Ingason, Yan & Zhu (2021) have circumvented this problem by freezing the Richardson number at unity once the critical regime is reached. This requires an artificial modification of the velocity which unfortunately fails to conserve fluxes. Recently, similarly to what has been done by Dhar, Das & Das (2020) for the flow of a thin water film, Haddad, Vaux, Varrall & Vauquelin (2022) proposed a method to face the singularity problem by introducing a discontinuity, similar to a hydraulic jump, to match the supercritical and subcritical regions. In their paper, they solved the equations numerically with an iterative procedure to determine the location of the jump for a given value of the domain length.

Nevertheless, and this is the purpose of the present paper, it is possible to go further and propose explicit solutions to the ET59 equations via purely analytical means. The methodology used to obtain these analytical solutions is in the same vein as that presented in Michaux & Vauquelin (2008) for the Morton *et al.* (1956) plume equations. Two cases are considered in this article: the first where the flow remains supercritical, and the second where the flow transitions from a supercritical to a subcritical regime.

3 Configuration and governing equations

As depicted in figure 1, we consider a fluid of density ρ_0 (lower than the density ρ_a of the ambient, at rest), injected horizontally from a plane nozzle of height h_0 with a velocity u_0 along a boundary of length L coincident with the horizontal x-axis. The flow at the injection is therefore characterised by the Richardson number $\mathrm{Ri}_0 = \Delta\rho_0 g h_0/\rho_a u_0^2$, which will be assumed to be less than unity (initially supercritical flow). As also shown in figure 1, the current has reached the end of the domain for a long period and therefore the steady state. For the sake of simplicity, the velocity and density profiles along the vertical z-axis will be considered uniform (top-hat assumption). So, at a given distance x from the injection, $u(x)$, $\rho(x)$ and $h(x)$ stand for the mean velocity, the mean density and the height of the current, respectively. As in ET59, the conservation equations for mass, momentum and buoyancy are established over an infinitesimal element of length dx of the current.

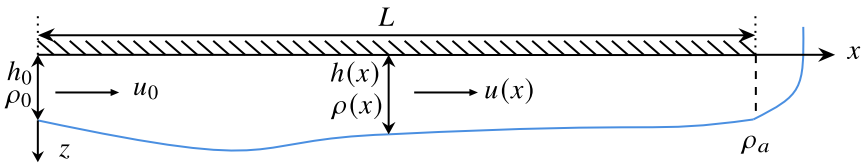

Figure 1: Schematic of the studied configuration.

In the Boussinesq approximation, these equations read as:

$$\frac{\mathrm{d}\,(u\,h)}{\mathrm{d}\,x} = E\,u, \tag{1}$$

$$\frac{\mathrm{d}\,(u^2\,h)}{\mathrm{d}\,x} = -C_d\,u^2 - \frac{1}{2}\frac{d}{\mathrm{d}\,x}\left(\eta\,g\,h^2\right), \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(\eta\,g\,u\,h\right) = 0, \tag{3}$$

with $\eta = \Delta\rho/\rho_a$ the density deficit, E the entrainment coefficient and C_d the drag coefficient.

The entrainment coefficient has been the subject of numerous experimental studies (the reader is referred to the extensive review of Fernando (1991)). Among the first who investigated this issue, we can mention ET59 and Lofquist (1960) who carried out experiments with fresh and salt water and showed a significant dependence of the entrainment coefficient on the Richardson number. These pioneering results, as well as those obtained in many subsequent studies, have been compiled in the paper of Christodoulou (1986) in order to propose universal laws for the entrainment in a gravity current. In the range of Ri = 0.1 - 10, he reports that the entrainment coefficient might be inversely proportional to the Richardson number as:

$$E = \frac{\alpha}{\mathrm{Ri}^n}, \tag{4}$$

where α is a constant equal to 0.002 and n is equal to 1. It is this model that will be considered in the calculations presented in this paper.

4 Analytical solutions

The conservation equations (1), (2), (3) and the entrainment model (4) can be combined to obtain the first-order derivatives of the height, velocity and density

deficit:

$$\frac{\mathrm{d}\,h}{dx} = \frac{\alpha}{2}\,\frac{4 + \mathrm{Ri}\,(\kappa - 2)}{\mathrm{Ri}\,(1 - \mathrm{Ri})}, \tag{5}$$

$$\frac{\mathrm{d}\,u}{dx} = -\frac{\alpha}{2}\,\frac{u}{h}\,\frac{2 + \kappa\,\mathrm{Ri}}{\mathrm{Ri}\,(1 - \mathrm{Ri})}, \tag{6}$$

$$\frac{\mathrm{d}\,\eta}{dx} = -\alpha\,\frac{\eta}{h}\,\frac{1}{\mathrm{Ri}}, \tag{7}$$

with $\kappa = 1 + (2\,C_d/\alpha)$. By combining equations (5), (6) and (7), we obtain the following relation which gives the first-order derivative of the local Richardson number $\mathrm{Ri}(x)$:

$$\frac{\mathrm{d}\,\mathrm{Ri}}{dx} = \frac{3\,\alpha}{2}\,\frac{1}{h}\,\frac{2 + \kappa\,\mathrm{Ri}}{(1 - \mathrm{Ri})}. \tag{8}$$

We notice that equations (5), (6) and (8) present a mathematical singularity when the Richardson number reaches unity. For an initially supercritical flow ($\mathrm{Ri}_0 < 1$), a quick look at the right-hand side of equation (8) reveals that Ri has to increase monotonically towards unity. With this in mind, two situations have to be considered: in the first one, the current remains supercritical from the injection until the exit of the domain, i.e. the Richardson number does not exceed unity and the equations (5), (6), (7) and (8) can be solved numerically without any difficulties. In the second one, the Richardson number reaches unity before the end of the domain, meaning that the flow has to transition from a supercritical to a subcritical regime, preventing solutions to be obtained by a straightforward integration.

In what follows, we will show how purely analytical solutions to the ET59 equations can be obtained, first for a current that remains supercritical, and then for a current that transitions from a supercritical to a subcritical regime before the end of the domain.

4.1 Supercritical flow over the whole domain

At first, by combining the equations (6) and (8), we easily show that the velocity $u(x)$ can be expressed as a function of the Richardson number $\mathrm{Ri}(x)$:

$$-\frac{1}{3}\frac{\mathrm{d}\,\mathrm{Ri}}{\mathrm{Ri}} = \frac{\mathrm{d}\,u}{u} \quad \Longrightarrow \quad \frac{u(x)}{u_0} = \left[\frac{\mathrm{Ri}_0}{\mathrm{Ri}(x)}\right]^{\frac{1}{3}}, \tag{9}$$

where the subscript 0 refers to values at the injection. We now combine the equations (7) and (8) to write:

$$\frac{1 - \mathrm{Ri}}{\mathrm{Ri}\,(2 + \kappa\,\mathrm{Ri})}\mathrm{d}\,\mathrm{Ri} = -\frac{3}{2}\,\frac{\mathrm{d}\,\eta}{\eta}, \tag{10}$$

which allows the density deficit $\eta(x)$ to be expressed as an explicit function of the Richardson number $\mathrm{Ri}(x)$:

$$\frac{\eta(x)}{\eta_0} = \left[\frac{\mathrm{Ri}_0}{\mathrm{Ri}(x)}\right]^{\frac{1}{3}} \left[\frac{2+\kappa\,\mathrm{Ri}(x)}{2+\kappa\,\mathrm{Ri}_0}\right]^{\frac{2+\kappa}{3\kappa}}. \tag{11}$$

Finally, from equations (3), (9) and (11), the height $h(x)$ of the current is expressed as a function of the Richardson number $\mathrm{Ri}(x)$ as:

$$\frac{h(x)}{h_0} = \left[\frac{\mathrm{Ri}(x)}{\mathrm{Ri}_0}\right]^{\frac{2}{3}} \left[\frac{2+\kappa\,\mathrm{Ri}_0}{2+\kappa\,\mathrm{Ri}(x)}\right]^{\frac{2+\kappa}{3\kappa}}. \tag{12}$$

The three relations (9), (11) and (12) allow the calculation of the longitudinal evolution of the characteristic variables of the current (velocity, density deficit and height) exclusively from the knowledge of the longitudinal evolution of the Richardson number. This latter can be determined by solving the equation (13) below, obtained from the equations (5) and (8):

$$\frac{\mathrm{d}\,\mathrm{Ri}}{\mathrm{d}\,x} = \frac{1}{\Lambda_0}\,\frac{(2+\kappa\,\mathrm{Ri})^{\frac{2+4\kappa}{3\kappa}}}{(1-\mathrm{Ri})\mathrm{Ri}^{\frac{2}{3}}}, \tag{13}$$

where the constant Λ_0, which has the dimension of a length, reads:

$$\Lambda_0 = \frac{2\,h_0(2+\kappa\,\mathrm{Ri}_0)^{\frac{2+\kappa}{3\kappa}}}{3\,\alpha\,\mathrm{Ri}_0^{\frac{2}{3}}}. \tag{14}$$

By integrating equation (13) from the injection to an abscissa x, it comes:

$$\int_{\mathrm{Ri}_0}^{\mathrm{Ri}(x)} \frac{(1-\zeta)\,\zeta^{\frac{2}{3}}}{(2+\kappa\,\zeta)^{\frac{2+4\kappa}{3\kappa}}}\,\mathrm{d}\,\zeta = \frac{1}{\Lambda_0}\int_0^x \mathrm{d}\,\xi. \tag{15}$$

We then introduce a universal function $F(X)$ defined by:

$$F(X) = \int_0^X \frac{(1-\zeta)\,\zeta^{\frac{2}{3}}}{(2+\kappa\,\zeta)^{\frac{2+4\kappa}{3\kappa}}}\,\mathrm{d}\,\zeta, \tag{16}$$

which allows the relation (15) to be rewritten as:

$$F[\mathrm{Ri}(x)] = \frac{x}{\Lambda_0} + F(\mathrm{Ri}_0). \tag{17}$$

In practice, once the values of α and C_d are set (typically $\alpha = 0.002$ according to Christodoulou (1986) and $C_d = 0.0065$, see for example Kunsch (2002)), the universal function F can be tabulated or plotted as it has been done in figure 2. The different steps of the calculation are as follows:

- From the injection conditions u_0, η_0 and h_0, Ri_0 and Λ_0 are first calculated.
- The value of $F(\mathrm{Ri}_0)$ is then determined with using equation (16) or graphically.
- For a given abscissa x, the quantity x/Λ_0 is added to this value, in order to obtain $F[Ri(x)]$, according to equation (17).
- As illustrated in figure 2, $\mathrm{Ri}(x)$ is then obtained graphically from $F[Ri(x)]$ via the universal function F.
- Finally, equations (9), (11), (12) allow the characteristic variables of the current, $u(x)$, $\eta(x)$ and $h(x)$, to be calculated.

This method applies if the Richardson number does not reach unity, i.e. if the flow remains supercritical. This condition is satisfied when the length L of the domain does not exceed a critical length L_c such as:

$$\int_{\mathrm{Ri}_0}^{1} \frac{(1-\zeta)\,\zeta^{\frac{2}{3}}}{(2+\kappa\,\zeta)^{\frac{2+4\kappa}{3\kappa}}}\,\mathrm{d}\,\zeta = \frac{1}{\Lambda_0}\int_0^{L_c}\mathrm{d}\,\xi \quad\Longrightarrow\quad L_c = \Lambda_0\,[F(1)-F(\mathrm{Ri}_0)]. \tag{18}$$

This relation is particularly interesting since it allows, from the knowledge of the injection conditions and the length of the domain, to know immediately whether or not the gravity current is likely to transition from a supercritical to a subcritical regime. Note that this critical length was discussed and numerically treated by Kostic & Parker (2007) in the particular case of a turbidity current developing along a boundary of finite length.

4.2 Supercritical flow with transition

We now consider the case where the length L of the domain is greater than the critical length L_c, i.e. the case when the flow transitions from a supercritical to a subcritical regime.

To take into account this transition in the equations of ET59, Haddad *et al.* (2022) proposed to introduce a mathematical discontinuity similar to a hydraulic jump at a location L_1, which leads the Richardson number and the height of the current to suddenly increase, as explicitly given by the well-known Bélanger equation:

$$\frac{h_2}{h_1} = \left(\frac{\mathrm{Ri}_2}{\mathrm{Ri}_1}\right)^{\frac{1}{3}} = \sigma(\mathrm{Ri}_1) \quad \text{with} \quad \sigma(X) = \frac{\sqrt{1+\frac{8}{X}}-1}{2}, \tag{19}$$

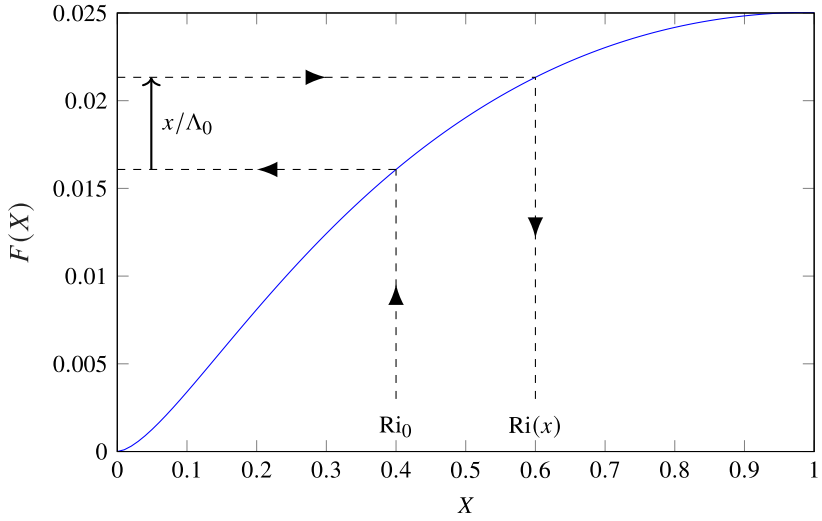

Figure 2: Graphical determination of the longitudinal evolution of the Richardson number via the universal function F (the function F has been drawn for $\alpha = 0.002$, $C_d = 0.0065$, i.e. $\kappa = 7.5$). The blue solid line represents the function F and the dashed lines are provided to illustrate the graphical procedure.

in which the subscript 1 is used for the quantities just upstream of the jump and the subscript 2 for these just downstream.

In this case, the theoretical problem of ET59 can be addressed by considering the connection between a supercritical flow (from Ri_0 to Ri_1 over a length L_1) and a subcritical flow (over a length $L - L_1$, from Ri_2 to the Richardson number at the exit of the domain, denoted Ri_L). Thus, based on the mathematical developments presented above, the location of the jump L_1, as well as the corresponding Richardson number just upstream Ri_1, can be found by solving the following system of equations:

$$\int_{\mathrm{Ri}_0}^{\mathrm{Ri}_1} \frac{(1-\zeta)\,\zeta^{\frac{2}{3}}}{(2+\kappa\,\zeta)^{\frac{2+4\,\kappa}{3\,\kappa}}}\,\mathrm{d}\zeta = \frac{1}{\Lambda_0}\int_0^{L_1}\mathrm{d}\xi \quad\Longrightarrow\quad F(\mathrm{Ri}_1) = \frac{L_1}{\Lambda_0} + F(\mathrm{Ri}_0), \tag{20}$$

$$\int_{\mathrm{Ri}_2}^{\mathrm{Ri}_L} \frac{(1-\zeta)\,\zeta^{\frac{2}{3}}}{(2+\kappa\,\zeta)^{\frac{2+4\,\kappa}{3\,\kappa}}}\,\mathrm{d}\zeta = \frac{1}{\Lambda_2}\int_{L_1}^{L}\mathrm{d}\xi \quad\Longrightarrow\quad F(\mathrm{Ri}_L) = \frac{L-L_1}{\Lambda_2} + F(\mathrm{Ri}_2), \tag{21}$$

where Λ_2 is given by:

$$\Lambda_2 = \frac{2\,h_2(2+\kappa\,\mathrm{Ri}_2)^{\frac{2+\kappa}{3\,\kappa}}}{3\,\alpha\mathrm{Ri}_2^{\frac{2}{3}}}. \tag{22}$$

At this stage, the system composed of equations (20) and (21) contains 5 unknowns,

namely Ri_1, L_1, Ri_2, Ri_L and h_2. First, concerning the Richardson number Ri_L at the exit, as explained by Henderson (1966), it should be close to unity for a subcritical current in an open channel. We will therefore consider hereafter that $\mathrm{Ri}_L = 1$. In addition, the use of the Bélanger equation (19) allows, on the one hand, to express Ri_2 as a function of Ri_1, and on the other hand, by using it in the relation (12), to obtain explicitly the value of the current height h_2 after the jump:

$$h_2 = h_0 \left(\frac{\mathrm{Ri}_1}{\mathrm{Ri}_0}\right)^{\frac{2}{3}} \left(\frac{2+\kappa\,\mathrm{Ri}_0}{2+\kappa\,\mathrm{Ri}_1}\right)^{\frac{2+\kappa}{3\kappa}} \sigma(\mathrm{Ri}_1). \tag{23}$$

After some algebra, we can show that Ri_1 is given by the following relation:

$$G(\mathrm{Ri}_1) = \frac{L}{\Lambda_0} + F(\mathrm{Ri}_0), \tag{24}$$

with G a universal function defined by:

$$G(X) = \left(\frac{2+\kappa\,X\,\sigma^3(X)}{2+\kappa\,X}\right)^{\frac{2+\kappa}{3\kappa}} \frac{F(1) - F(X\,\sigma^3(X))}{\sigma(X)} + F(X). \tag{25}$$

Practically, for a fixed value of κ, the function G, similarly to the function F, can be tabulated or plotted as shown in figure 3. From the initial conditions, we first calculate Ri_0, Λ_0 and $F(\mathrm{Ri}_0)$. Then, we add to $F(\mathrm{Ri}_0)$ the quantity L/Λ_0 which allows Ri_1 to be determined graphically, according to equation (24).

Once Ri_1 is known, Ri_2 is calculated with the Bélanger equation (19), L_1 by equation (20), h_2 by equation (23) and Λ_2 by equation (22). Finally, according to equation (15), the longitudinal evolution of the Richardson number in the subcritical region can be obtained by the following equation:

$$F[\mathrm{Ri}(x)] = \frac{x - L_1}{\Lambda_2} + F(\mathrm{Ri}_2). \tag{26}$$

The graphical determination can be done by means of the graph in figure 4 on which the universal function F is now plotted in the range of Richardson numbers between 1 and 10, corresponding to subcritical flows.

Once $\mathrm{Ri}(x)$ is known over the whole domain, the characteristic variables $u(x)$, $h(x)$ and $\eta(x)$ of the current are calculated immediately from (9), (11) and (12) in the supercritical region, and in the subcritical region with the following relations:

$$\frac{u(x)}{u_2} = \left[\frac{\mathrm{Ri}_2}{\mathrm{Ri}(x)}\right]^{\frac{1}{3}}, \quad \frac{\eta(x)}{\eta_2} = \left[\frac{\mathrm{Ri}_2}{\mathrm{Ri}(x)}\right]^{\frac{1}{3}} \left[\frac{2+\kappa\,\mathrm{Ri}(x)}{2+\kappa\,\mathrm{Ri}_2}\right]^{\frac{2+\kappa}{3\kappa}}$$
$$\text{and} \quad \frac{h(x)}{h_2} = \left[\frac{\mathrm{Ri}(x)}{\mathrm{Ri}_2}\right]^{\frac{2}{3}} \left[\frac{2+\kappa\,\mathrm{Ri}_2}{2+\kappa\,\mathrm{Ri}(x)}\right]^{\frac{2+\kappa}{3\kappa}}. \tag{27}$$

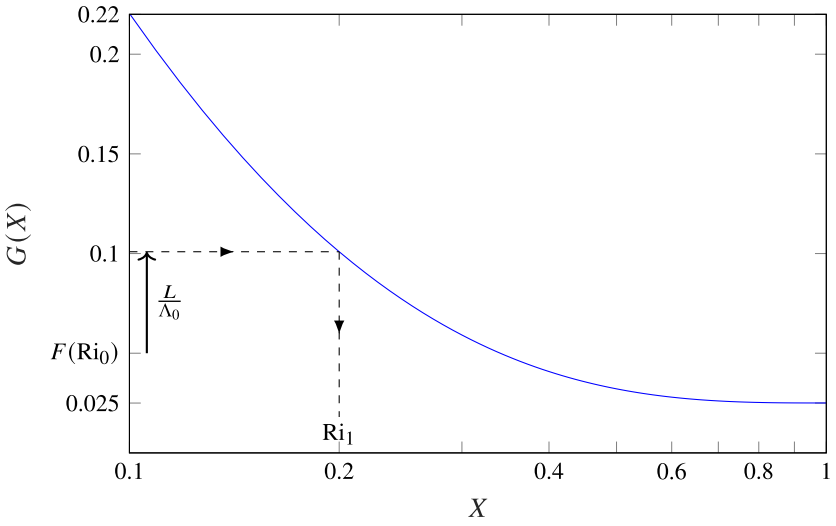

Figure 3: Graphical determination of the Richardson number just upstream of the jump Ri_1 via the universal function G (the $x-$axis is logarithmic and the function G has been calculated for $\alpha = 0.002$, $C_d = 0.0065$, i.e. $\kappa = 7.5$). The blue solid line represents the function G and the dashed lines are provided to illustrate the graphical procedure.

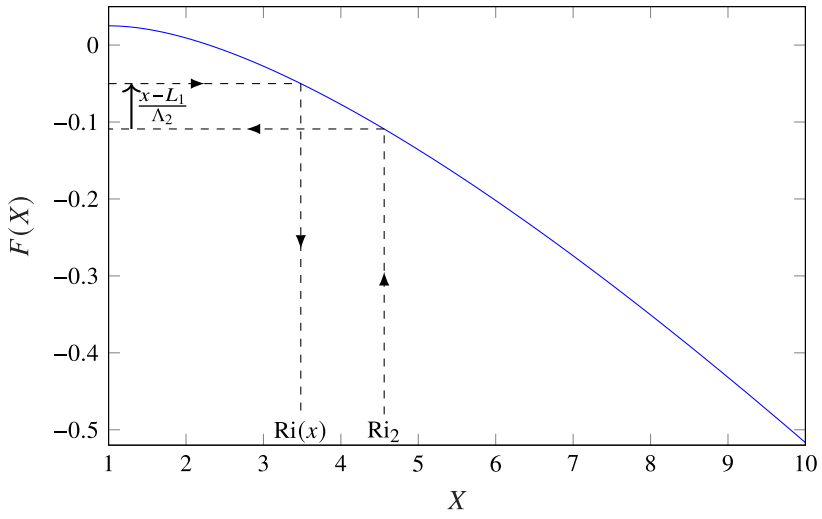

Figure 4: Graphical determination of the longitudinal evolution of the Richardson number in the subcritical region via the universal function F (the function F has been drawn for $\alpha = 0.002$, $C_d = 0.0065$, i.e. $\kappa = 7.5$). The blue solid line represents the function F and the dashed lines are provided to illustrate the graphical procedure.

5 Conclusions and discussions

This paper reports an analytical method to solve the equations of Ellison & Turner (1959) describing the longitudinal evolution of a gravity current at steady state. The buoyant fluid that forms the current is continuously injected from a plane nozzle along a horizontal rigid boundary of finite length.

First, expressions of the primary variables of the current (velocity u, height h and density deficit η) as explicit functions of the Richardson number $\mathrm{Ri}(x)$ are established. Then, the longitudinal evolution of the Richardson number is obtained from a universal function F (that depends only on the entrainment and drag coefficients). This universal function is given by an indefinite integral which can be tabulated or represented graphically. Here, we have illustrated the method by means of a graphical representation which, for prescribed injection conditions, allows the Richardson number value to be determined readily at any abscissa x along the horizontal boundary.

The ET59 equations presenting a mathematical singularity when $Ri = 1$, the developed method applies therefore easily as long as the flow remains supercritical ($Ri < 1$) on the whole domain. Note that from the knowledge of the injection conditions and the length of the domain, it is possible to determine theoretically thanks to the universal function F whether the flow remains supercritical or should transition to a subcritical state ($Ri > 1$).

If there is a transition (i.e. if the Richardson number reaches unity before the exit of the domain), the two regimes (supercritical and subcritical) can coexist by introducing in the ET59 equations a mathematical discontinuity similar to a hydraulic jump. For the sake of simplicity, we have modelled this jump using the Bélanger relation. In practice, the location of the jump is obtained from the injection conditions and a second universal function G. The evolution of the primary variables is then obtained for each zone (upstream and downstream of the jump) from the first universal function F.

Moreover, this method can be extended for various entrainement laws proposed by Christodoulou (1986) (i.e. $n = 0$ and $n = 1/2$), allowing a greater flexibility of the model as well as a wider range of Richardson number.

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