



**HAL**  
open science

# Analytical solutions for long-time steady state Boussinesq gravity currents flowing along a horizontal boundary of finite length

Safir Haddad, Samuel Vaux, Kevin Varrall, Olivier Vauquelin

► **To cite this version:**

Safir Haddad, Samuel Vaux, Kevin Varrall, Olivier Vauquelin. Analytical solutions for long-time steady state Boussinesq gravity currents flowing along a horizontal boundary of finite length. *Physical Review Fluids*, 2024, 9 (7), pp.074803. 10.1103/PhysRevFluids.9.074803 . irsn-04630395

**HAL Id: irsn-04630395**

**<https://irsn.hal.science/irsn-04630395v1>**

Submitted on 1 Jul 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Analytical solutions for long-time steady state Boussinesq gravity currents flowing along a horizontal boundary of finite length

Safir Haddad,<sup>\*</sup> Samuel Vaux,<sup>†</sup> Kevin Varrall,<sup>‡</sup> and Olivier Vauquelin<sup>§</sup>  
<sup>1</sup>*Institut de Radioprotection et de Sûreté Nucléaire (IRSN), PSN-RES,  
SA2I, LIE, Cadarache, 13115 Saint-Paul-lez-Durance, France and*  
<sup>2</sup>*Aix-Marseille Université (AMU), Laboratoire IUSTI,  
CNRS UMR 7343, 5 Rue Enrico Fermi, 13453 Marseille Cedex, France*

(Dated: July 1, 2024)

## Abstract

This paper presents analytical solutions for a steady turbulent miscible gravity current flowing along a horizontal rigid boundary of finite length into a quiescent uniform environment. These solutions are obtained from the governing equations (mass, momentum and buoyancy) originally proposed by Ellison & Turner [*J. Fluid. Mech.*, **6**, 423 (1959)] for a buoyant layer of fluid in the Boussinesq approximation. For a constant drag coefficient  $C_d$  and the specific entrainment law  $E \propto \text{Ri}^{-1}$ ,  $\text{Ri}$  being the local Richardson number, we first derived a system of coupled ordinary differential equations describing the longitudinal evolution of the velocity  $u$ , the height  $h$ , the density deficit  $\eta$  and the Richardson number  $\text{Ri}$  of the current. For an initially supercritical flow ( $\text{Ri}_0 < 1$ ), explicit relations are found for  $u(x)$ ,  $h(x)$  and  $\eta(x)$  solely as a function of the Richardson number  $\text{Ri}(x)$ . The longitudinal evolution of the Richardson number is then theoretically obtained from a universal function  $F$  which can be tabulated and, as in the present paper, also plotted. The function  $F$  allows us to determine (and only from the knowledge of the boundary conditions at the source) whether the flow remains supercritical over the whole length of the rigid boundary, or might transit towards a subcritical state ( $\text{Ri} > 1$ ). In this latter case, the mathematical resolution is modified by including a discontinuity similar to a hydraulic jump. The location and amplitude of this discontinuity are calculated from an additional universal function  $G$  and the injection conditions. The method is finally extended to provide analytical solutions for other classical entrainment laws.

---

<sup>\*</sup> safir.haddad@univ-amu.fr<sup>1,2</sup>

<sup>†</sup> samuel.vaux@irsn.fr<sup>1</sup>

<sup>‡</sup> kevin.varrall@univ-amu.fr<sup>2</sup>

<sup>§</sup> olivier.vauquelin@univ-amu.fr<sup>2</sup>

## I. INTRODUCTION

A gravity current is a canonical flow that occurs when a light (heavy) fluid propagates into a heavier (lighter) ambient fluid along a rigid boundary. This flow can involve immiscible or miscible fluids. In the latter case, the current engulfs the surrounding fluid in a process called entrainment, resulting in a longitudinal evolution of the current mass flow rate.

Gravity currents arise in many environmental flows such as katabatic winds (Manins and Sawford [1]), oceanic deep currents (Cenedese and Adduce [2]) or turbidity currents (Meiburg and Kneller [3]), to name but a few. They may also appear in hazardous situations such as oil spreading on the sea (Hoult [4]) or fire-induced smoke propagation (Alpert [5]).

Because of both their ubiquity and academic interest, gravity currents have been widely studied. The pioneering works of Von Kármán [6] and later Benjamin [7] were the first to tackle this flow theoretically. One of the main objectives of these works was to determine the dynamics of the current head during the propagation phase. Several authors have subsequently addressed the transient evolution of this flow by proposing models for gravity currents resulting from a fixed-volume release using experimental (Huppert and Simpson [8], Rottman and Simpson [9], Lowe, Rottman, and Linden [10]) numerical (Birman, Battandier, Meiburg, and Linden [11], Bonometti, Ungarish, and Balachandar [12]) and theoretical means (Shin, Dalziel, and Linden [13]), as well as from a fixed-flux release with experiments (Longo, Ungarish, Di Federico, Chiapponi, and Addona [14], Sher and Woods [15], Martin, Negretti, Ungarish, and Zemach [16]), numerical simulations (Shringarpure, Lee, Ungarish, and Balachandar [17], Hogg, Nasr-Azadani, Ungarish, and Meiburg [18]) and theory (Johnson and Hogg [19], Ungarish [20]).

In contrast to the transient phase of gravity currents, that has been addressed in a substantial number of studies, the steady phase (i.e. after a long time subsequent to the reaching of the exit of the rigid boundary by the flow) remains little investigated so far. In their seminal article, Ellison and Turner [21] (hereafter referred to as ET59) have developed a theoretical model for a fixed-flux steady gravity current based on the conservation equations for mass, momentum and buoyancy. Through a system of coupled differential equations, their model allows the evolution of the three variables of the current (velocity, thickness and density deficit) to be calculated along the longitudinal propagation  $x$ -axis. Similarly to Morton, Taylor, and Turner [22] for turbulent plumes, they introduced an entrainment

coefficient  $E$  to quantify the amount of ambient fluid entrained into the current. After some algebraic manipulations, the Richardson number  $Ri$  naturally appears in their equations. It is defined as:

$$Ri = \frac{\eta g h}{u^2}, \quad (1)$$

with  $\eta = |\rho_a - \rho|/\rho_a$  the density deficit,  $\rho_a$  the density of the ambient,  $g$  the gravitational acceleration and  $h$ ,  $\rho$  and  $u$ , the thickness, the density and velocity of the current, respectively. As explained in ET59, the Richardson number, which characterises the local stability of the current, allows three different flow regimes to be identified: the supercritical regime when  $Ri < 1$ , the subcritical regime when  $Ri > 1$  and a so-called critical regime when  $Ri = 1$ .

A particular feature of this model (as shown in section III B), which can also be found in some hydraulic problems (Wilkinson and Wood [23]), is the presence of a mathematical singularity which raises an additional problem when  $Ri = 1$ .

In their attempt to solve the ET59 equations, Guo, Li, Ingason, Yan, and Zhu [24] have circumvented this problem by freezing the Richardson number at unity once the critical condition is reached. This requires an artificial modification of the velocity which unfortunately fails to conserve fluxes. Recently, similarly to what has been done by Dhar, Das, and Das [25] for the flow of a thin water film, Haddad, Vaux, Varrall, and Vauquelin [26] proposed a method to face the singularity problem by introducing a discontinuity, similar to a hydraulic jump, to match the supercritical and subcritical regions. In their paper, they solved the equations numerically with an iterative procedure to determine the location and amplitude of the jump for given values of the domain length and the initial release conditions. The authors also compared the results provided by solving the equations of ET59 extended in the non-Boussinesq configuration with the results obtained from LES simulations. They found good agreement between the simulations and theory, particularly for fully supercritical flows.

Nevertheless, and this is the purpose of the present paper, it is possible to go further and propose explicit solutions to the ET59 equations via purely analytical means. The methodology used to obtain these analytical solutions is in the same vein as that presented in Michaux and Vauquelin [27] for the Morton *et al.* [22] plume equations.

The paper is set out as follows. The studied configuration and the governing equations are presented in Section II. Analytical solutions to these equations are proposed in Section III for a specific entrainment law ( $E \propto Ri^{-1}$ ) in the case of a supercritical flow, but also for

a flow with a regime change (supercritical to subcritical). The method is finally extended to other classical entrainment laws in Section IV in order to apply the modeling to a wider range of real-world cases. Conclusions are drawn in Section V.

## II. CONFIGURATION AND GOVERNING EQUATIONS

As depicted in figure 1, we consider a fluid of density  $\rho_0$  (lower than the density  $\rho_a$  of the ambient, at rest), injected horizontally from a plane nozzle of height  $h_0$  with a velocity  $u_0$  along a boundary of length  $L$  coincident with the horizontal  $x$ -axis. The flow at the injection is therefore characterised by its Richardson number  $\text{Ri}_0 = \eta_0 g h_0 / u_0^2$ , which will be assumed to be less than unity (the flow is then initially supercritical). We consider that the current has reached the end of the domain for a long period and therefore a steady state. For the sake of simplicity, the velocity and density profiles along the vertical  $z$ -axis will be considered uniform (top-hat assumption). Also, shape factors, used by ET59 to take into account the deviation between the real profiles and the top-hat profiles of the velocity and the density, are set to unity, even if they could be adjusted to non-unity values without preventing the algebraic development presented in this article. So, at a given distance  $x$  from the injection,  $u(x)$ ,  $\eta(x)$  and  $h(x)$  stand for the top-hat velocity, the top-hat density deficit and the height of the current, respectively. As in ET59, the conservation equations for mass, momentum and buoyancy are established over an infinitesimal element of length  $dx$  of the current. In the Boussinesq approximation, these equations read as:

$$\frac{d(uh)}{dx} = E u, \quad (2)$$

$$\frac{d(u^2 h)}{dx} = -C_d u^2 - \frac{1}{2} \frac{d}{dx} (\eta g h^2), \quad (3)$$

$$\frac{d}{dx} (\eta g u h) = 0, \quad (4)$$

with  $E$  the entrainment coefficient and  $C_d$  the drag coefficient. The terms on the right-hand side of equation (3) represent respectively the turbulent basal drag and the pressure force associated with the change of height and density of the current.

In contrast to the entrainment coefficient for a turbulent plume, which can be considered as a constant, the entrainment coefficient for a gravity current depends on the stability of the flow and therefore on the Richardson number. This is why this coefficient has been the subject of numerous studies. Papers by ET59 and Lofquist [28] were some of the first to

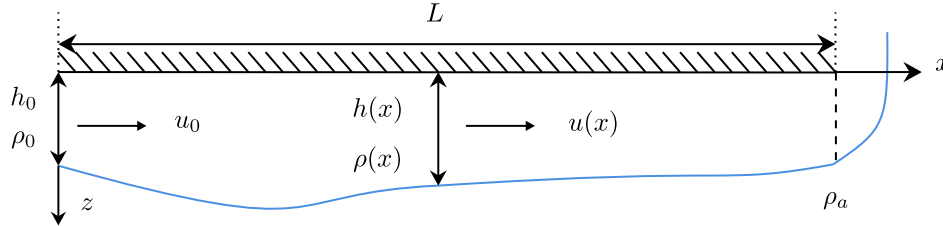


FIG. 1: Schematic of the studied configuration.

address this issue experimentally with salt water experiments. They highlighted the marked dependence of the entrainment coefficient on the Richardson number and proposed laws in which  $E$  is a function of  $Ri$  only. Other significant contributions include the experimental works of Parker, Fukushima, and Pantin [29], Dallimore, Imberger, and Ishikawa [30] and Wells, Cenedese, and Caulfield [31]. In addition, the reader is referred to the extensive review of Fernando [32] and Chowdhury and Testik [33] and also to the article of Christodoulou [34] in which experimental results from the literature are compiled to provide power laws for several ranges of Richardson number under the form:

$$E = \frac{\alpha}{Ri^n}, \quad (5)$$

where  $\alpha$  and  $n$  are constants. This modeling can only be used for steady configurations in which the entrainment process concerns a quasi-horizontal surface. In the unsteady regime, and as explained and illustrated by Nogueira, Adduce, Alves, and Franca [35] and Sher and Woods [15], the flow morphology is more complex and requires the use of specific entrainment models for the head, the body and even the tail.

The drag coefficient has also been investigated in the literature, and it is not uncommon to choose it as being constant (Hogg and Woods [36], Baines [37]). We have also made this assumption in the following sections of this paper.

In the next section, when presenting the approach to analytically solving the ET59 equations, we will use, in addition to a constant drag coefficient, the submodel  $E = \alpha/Ri$  proposed by Christodoulou [34] for the range  $0.1 < Ri < 10$ . The advantage of selecting this submodel is that it theoretically encompasses a broad range of the super- and sub-critical flows with a unique entrainment model.

### III. ANALYTICAL SOLUTIONS

By combining conservation equations (2), (3), (4) and the entrainment submodel (5) with  $n = 1$ , we obtain the first-order derivatives of the height, velocity and density deficit as:

$$\frac{d h}{d x} = \frac{\alpha}{2} \frac{4 + \text{Ri}(\kappa - 2)}{\text{Ri}(1 - \text{Ri})}, \quad (6)$$

$$\frac{d u}{d x} = -\frac{\alpha}{2} \frac{u}{h} \frac{2 + \kappa \text{Ri}}{\text{Ri}(1 - \text{Ri})}, \quad (7)$$

$$\frac{d \eta}{d x} = -\alpha \frac{\eta}{h} \frac{1}{\text{Ri}}, \quad (8)$$

where  $\kappa$  is a constant equal to  $1 + 2C_d/\alpha$ .

Additionally, the first-order derivative of the local Richardson number  $\text{Ri}(x)$  can be obtained by combining equations (1), (6), (7) and (8). It reads:

$$\frac{d \text{Ri}}{d x} = \frac{3 \alpha}{2} \frac{1}{h} \frac{2 + \kappa \text{Ri}}{(1 - \text{Ri})}. \quad (9)$$

We notice that equations (6), (7) and (9) present a mathematical singularity when the Richardson number reaches unity. For an initially supercritical flow ( $\text{Ri}_0 < 1$ ), a quick look at the right-hand side of equation (9) reveals that  $\text{Ri}$  has to increase monotonically. With this in mind, two situations have to be considered. In the first one, the current remains supercritical from the injection until the exit of the domain, i.e. the Richardson number never exceeds unity. The set of equations (6), (7), (8) and (9) can be solved without difficulty. In the second one, the Richardson number reaches unity before the end of the domain, meaning that the flow has to transition from a supercritical to a subcritical regime, preventing solutions to be obtained by a straightforward integration. This issue was recently addressed by Haddad *et al.* [26] and discussed by Ungarish [38].

In both cases (the flow remains supercritical or becomes subcritical before the end of the domain), analytical solutions can be obtained. This is the purpose of the following subsections.

#### A. Initially supercritical flow remaining supercritical

At first, by combining the equations (7) and (9), we easily show that the velocity  $u(x)$  can be expressed as a function of the Richardson number  $\text{Ri}(x)$ :

$$-\frac{1}{3} \frac{d \text{Ri}}{\text{Ri}} = \frac{d u}{u} \implies \frac{u(x)}{u_0} = \left[ \frac{\text{Ri}_0}{\text{Ri}(x)} \right]^{\frac{1}{3}}, \quad (10)$$

where the subscript 0 refers to values at the injection. We now combine the equations (8) and (9) to write:

$$\frac{1 - \text{Ri}}{\text{Ri}(2 + \kappa \text{Ri})} d \text{Ri} = -\frac{3}{2} \frac{d \eta}{\eta}, \quad (11)$$

which allows the density deficit  $\eta(x)$  to be expressed explicitly as a function of the Richardson number  $\text{Ri}(x)$ :

$$\frac{\eta(x)}{\eta_0} = \left[ \frac{\text{Ri}_0}{\text{Ri}(x)} \right]^{\frac{1}{3}} \left[ \frac{2 + \kappa \text{Ri}(x)}{2 + \kappa \text{Ri}_0} \right]^{\frac{2+\kappa}{3\kappa}}. \quad (12)$$

Finally, given the buoyancy conservation (4) and equations (10) and (12), we immediately obtain for the height  $h(x)$  of the current:

$$\frac{h(x)}{h_0} = \left[ \frac{\text{Ri}(x)}{\text{Ri}_0} \right]^{\frac{2}{3}} \left[ \frac{2 + \kappa \text{Ri}_0}{2 + \kappa \text{Ri}(x)} \right]^{\frac{2+\kappa}{3\kappa}}. \quad (13)$$

The three relations (10), (12) and (13) allow the calculation of the longitudinal evolution of the characteristic variables of the current (velocity, density deficit and height) exclusively from the knowledge of the longitudinal evolution of the Richardson number. This latter can be determined by solving the equation (14) below, obtained from equations (9) and (13):

$$\frac{d \text{Ri}}{d x} = \frac{1}{\Lambda_0} \frac{(2 + \kappa \text{Ri})^{\frac{2+4\kappa}{3\kappa}}}{(1 - \text{Ri})\text{Ri}^{\frac{2}{3}}}, \quad (14)$$

where the constant  $\Lambda_0$ , which has the dimension of a length, reads:

$$\Lambda_0 = \frac{2 h_0 (2 + \kappa \text{Ri}_0)^{\frac{2+\kappa}{3\kappa}}}{3 \alpha \text{Ri}_0^{\frac{2}{3}}}. \quad (15)$$

By integrating equation (14) from the injection to an abscissa  $x$ , it becomes:

$$\int_{\text{Ri}_0}^{\text{Ri}(x)} \frac{(1 - \zeta) \zeta^{\frac{2}{3}}}{(2 + \kappa \zeta)^{\frac{2+4\kappa}{3\kappa}}} d \zeta = \frac{1}{\Lambda_0} \int_0^x d \xi. \quad (16)$$

We then introduce a universal function  $F(X)$  defined by:

$$F(X) = \int_0^X \frac{(1 - \zeta) \zeta^{\frac{2}{3}}}{(2 + \kappa \zeta)^{\frac{2+4\kappa}{3\kappa}}} d \zeta, \quad (17)$$

which allows the relation (16) to be rewritten as:

$$F[\text{Ri}(x)] = \frac{x}{\Lambda_0} + F(\text{Ri}_0). \quad (18)$$

In practice, once the values of  $\alpha$  and  $C_d$  are set, the different steps of the calculation are as follows:



- From the injection conditions  $u_0$ ,  $\eta_0$  and  $h_0$ ,  $\text{Ri}_0$  and  $\Lambda_0$  are first calculated.
- The value of  $F(\text{Ri}_0)$  is then determined with using equation (17).
- For a given abscissa  $x$ , the quantity  $x/\Lambda_0$  is added to this value, in order to obtain  $F[\text{Ri}(x)]$ , according to equation (18).
- $\text{Ri}(x)$  is then obtained from  $F[\text{Ri}(x)]$  via the inverse of the universal function  $F$ .
- Finally, equations (10), (12), (13) allow the characteristic variables of the current,  $u(x)$ ,  $\eta(x)$  and  $h(x)$ , to be calculated.

Although the function  $F$  can be calculated numerically without difficulty, we propose in figure 2 a graphical illustration of the method for several values of the constant  $\kappa$ .

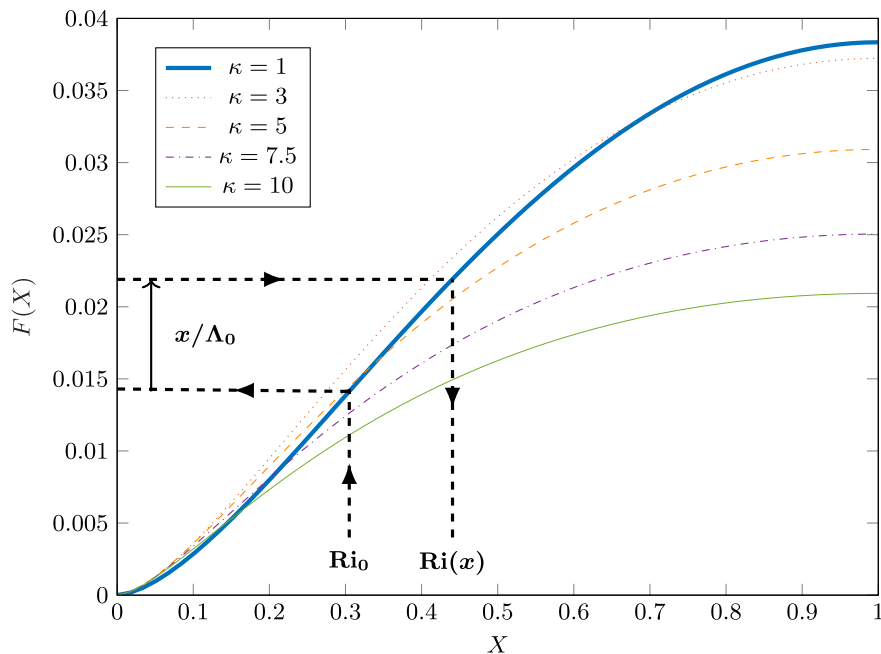


FIG. 2: Illustration of the graphical determination of the longitudinal evolution of the Richardson number via the universal function  $F$  for several values of  $\kappa$ . The dashed lines are provided to illustrate the graphical procedure for  $\kappa = 1$ .

Note that, from relation (18), it is possible to determine the critical length  $L_c$  at which the flow theoretically reaches the critical condition (i.e.  $\text{Ri} = 1$ ). We therefore obtain:

$$L_c = \Lambda_0 [F(1) - F(\text{Ri}_0)]. \quad (19)$$

This relation is particularly interesting since it allows, from the knowledge of the injection conditions and the length of the domain, to know immediately whether or not the gravity current is likely to transition from a supercritical to a subcritical regime. This issue was discussed by Kostic and Parker [39] in the particular case of a turbidity current developing along a boundary of finite length.

### B. Initially supercritical flow becoming subcritical

We now consider the case where the length  $L$  of the domain is greater than the critical length  $L_c$ , i.e. the case when the flow transitions from a supercritical to a subcritical regime.

To take into account this transition in the equations of ET59, Haddad *et al.* [26] introduced a mathematical discontinuity similar to a hydraulic jump at a location  $L_1$ , which leads the Richardson number and the height of the current to suddenly increase to match the subcritical regime.

Assuming that the density deficit in the current does not change on either side of the discontinuity, the governing equations (mass and momentum) of the jump are:

$$u_1 h_1 = u_2 h_2, \tag{20}$$

$$u_1^2 h_1 + \frac{\eta_1 g h_1^2}{2} = u_2^2 h_2 + \frac{\eta_2 g h_2^2}{2}, \tag{21}$$

in which the subscripts 1 and 2 are used for the quantities just upstream and just downstream of the jump, respectively. By combining these two equations, we obtain the jump relation, also known as Bélanger equation (or Rankine-Hugoniot condition):

$$\frac{h_2}{h_1} = \left( \frac{\text{Ri}_2}{\text{Ri}_1} \right)^{\frac{1}{3}} = \sigma(\text{Ri}_1), \tag{22}$$

with:

$$\sigma(X) = \frac{\sqrt{1 + \frac{8}{X}} - 1}{2}. \tag{23}$$

The theoretical problem of ET59 can be addressed by considering the connection between a supercritical flow (from  $\text{Ri}_0$  to  $\text{Ri}_1$  over a length  $L_1$ ) and a subcritical flow (over a length  $L - L_1$ , from  $\text{Ri}_2$  to the Richardson number at the exit of the domain, denoted  $\text{Ri}_L$ ), as illustrated in figure 3.

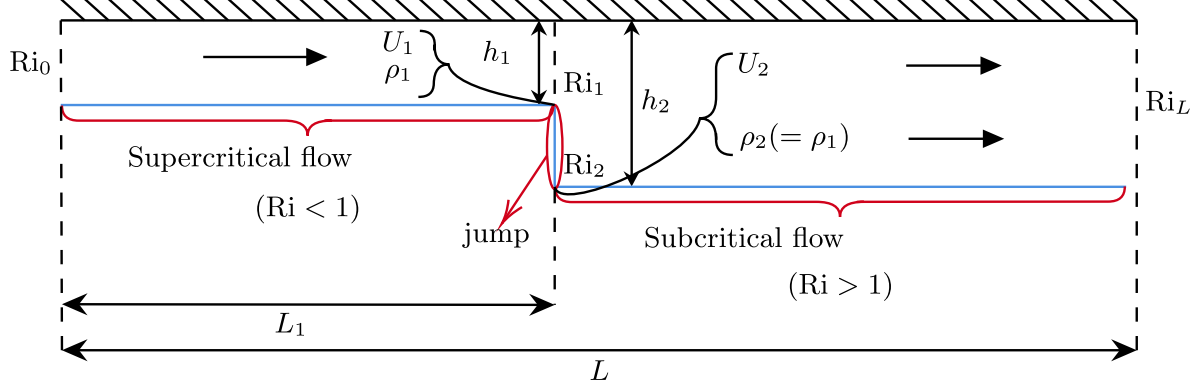


FIG. 3: Representation of a gravity current including an idealized transition between the supercritical and the subcritical flows.

Thus, based on the mathematical developments presented above, the location of the jump  $L_1$ , as well as the corresponding Richardson number  $Ri_1$  just upstream, can be found by solving the following system of equations:

$$F(Ri_1) = \frac{L_1}{\Lambda_0} + F(Ri_0), \quad (24)$$

$$F(Ri_L) = \frac{L - L_1}{\Lambda_2} + F(Ri_2), \quad (25)$$

where  $\Lambda_2$  is given by:

$$\Lambda_2 = \frac{2 h_2 (2 + \kappa Ri_2)^{\frac{2+\kappa}{3\kappa}}}{3 \alpha Ri_2^{\frac{2}{3}}}. \quad (26)$$

At this stage, the system composed of equations (24) and (25) contains 5 unknowns, namely  $Ri_1$ ,  $L_1$ ,  $Ri_2$ ,  $Ri_L$  and  $h_2$ . First, concerning the Richardson number  $Ri_L$  at the exit, as explained by Henderson [40], it should be close to unity for a subcritical current in an open channel. We will therefore consider hereafter that  $Ri_L = 1$ . In addition, the use of the Bélanger equation (22) allows, on the one hand, to express  $Ri_2$  as a function of  $Ri_1$ , and on the other hand, by using it in the relation (13), to obtain explicitly the value of the current height  $h_2$  after the jump:

$$h_2 = h_0 \left( \frac{Ri_1}{Ri_0} \right)^{\frac{2}{3}} \left( \frac{2 + \kappa Ri_0}{2 + \kappa Ri_1} \right)^{\frac{2+\kappa}{3\kappa}} \sigma(Ri_1). \quad (27)$$

After some algebra, we can show that  $Ri_1$  is given by the following relation:

$$G(Ri_1) = \frac{L}{\Lambda_0} + F(Ri_0), \quad (28)$$

with  $G$  a universal function defined by:

$$G(X) = \left( \frac{2 + \kappa X \sigma^3(X)}{2 + \kappa X} \right)^{\frac{2+\kappa}{3\kappa}} \frac{F(1) - F[X \sigma^3(X)]}{\sigma(X)} + F(X). \quad (29)$$

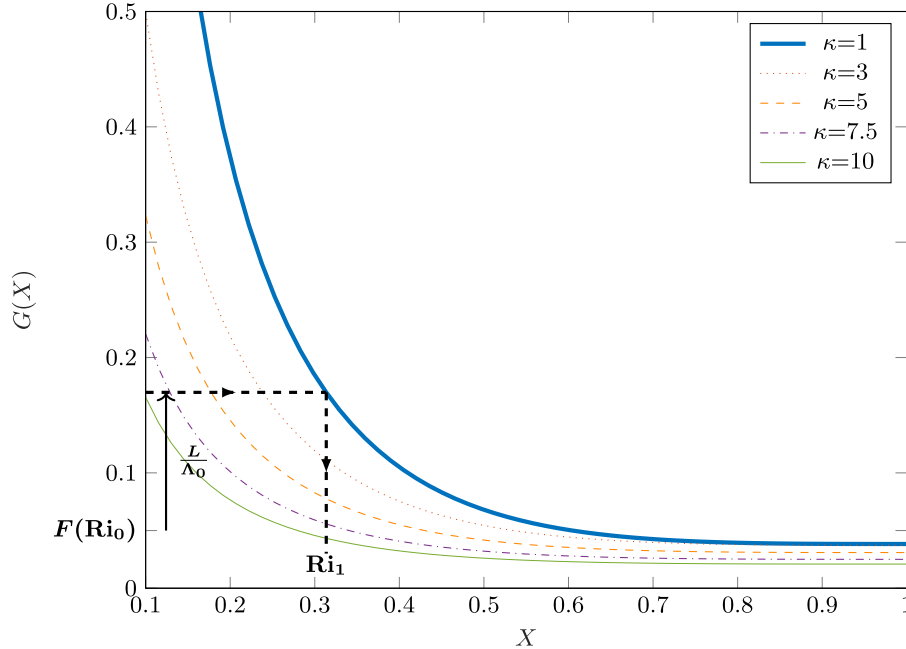


FIG. 4: Illustration of the graphical determination of the Richardson number  $Ri_1$  just upstream of the jump via the universal function  $G$  for several values of  $\kappa$ . The dashed lines are provided to illustrate the graphical procedure for  $\kappa = 1$ .

Practically, for a fixed value of  $\kappa$ , we first calculate  $Ri_0$ ,  $\Lambda_0$  and  $F(Ri_0)$ . Then, we add to  $F(Ri_0)$  the quantity  $L/\Lambda_0$  which allows  $Ri_1$  to be calculated with the equation (28) and the inverse of the function  $G$ . A graphical illustration of the method is proposed in figure 4.

Once  $Ri_1$  is known,  $Ri_2$  is calculated with the Bélanger equation (22),  $L_1$  by equation (24),  $h_2$  by equation (27) and  $\Lambda_2$  by equation (26). Finally, according to equation (16), the longitudinal evolution of the Richardson number in the subcritical region can be obtained graphically (see figure 5), or numerically from the following equation:

$$F[Ri(x)] = \frac{x - L_1}{\Lambda_2} + F(Ri_2). \quad (30)$$

Once  $Ri(x)$  is known over the whole domain, the characteristic variables  $u(x)$ ,  $h(x)$  and  $\eta(x)$  of the current are calculated immediately from (10), (12) and (13) in the supercritical

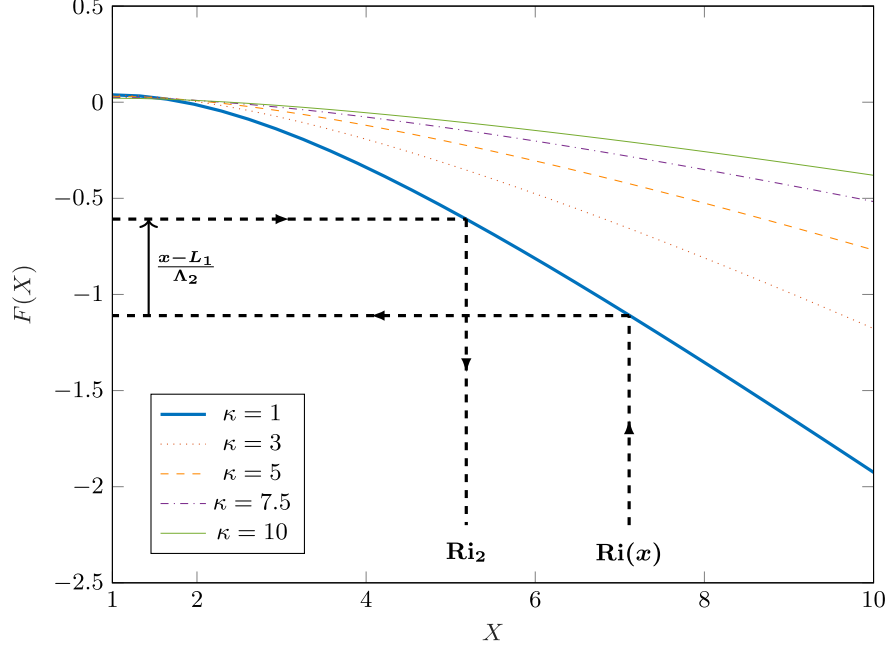


FIG. 5: Illustration of the graphical determination of the longitudinal evolution of the Richardson number downstream of the jump (subcritical regime) via the universal function  $F$  for several values of  $\kappa$ . The dashed lines are provided to illustrate the graphical procedure for  $\kappa = 1$ .

region, and in the subcritical region with the following relations:

$$\frac{u(x)}{u_2} = \left[ \frac{\text{Ri}_2}{\text{Ri}(x)} \right]^{\frac{1}{3}}, \quad \frac{\eta(x)}{\eta_2} = \left[ \frac{\text{Ri}_2}{\text{Ri}(x)} \right]^{\frac{1}{3}} \left[ \frac{2 + \kappa \text{Ri}(x)}{2 + \kappa \text{Ri}_2} \right]^{\frac{2+\kappa}{3\kappa}}$$

$$\text{and} \quad \frac{h(x)}{h_2} = \left[ \frac{\text{Ri}(x)}{\text{Ri}_2} \right]^{\frac{2}{3}} \left[ \frac{2 + \kappa \text{Ri}_2}{2 + \kappa \text{Ri}(x)} \right]^{\frac{2+\kappa}{3\kappa}}. \quad (31)$$

#### IV. ANALYTICAL SOLUTIONS FOR OTHER ENTRAINMENT LAWS

The methodology for obtaining analytical solutions presented in the previous section considered the specific entrainment model  $E = \alpha/\text{Ri}$ . In this section we propose to extend the approach for other entrainment laws from the literature. These laws are often associated with particular flow regimes, as it will be specified.

### A. No entrainment

A null entrainment coefficient can be associated with a strongly buoyant gravity current for which buoyancy forces inhibit the mixing such as long-distance submarine currents with strong stratification. The no-entrainment configuration can also correspond to hydraulic or immiscible flows.

In this case, the density deficit turns to be a constant of the problem and the two primary variables  $u(x)$  and  $h(x)$  are immediately expressed as a function of the local Richardson number as:

$$\frac{u(x)}{u_0} = \left[ \frac{\text{Ri}_0}{\text{Ri}(x)} \right]^{\frac{1}{3}} \quad \text{and} \quad \frac{h(x)}{h_0} = \left[ \frac{\text{Ri}(x)}{\text{Ri}_0} \right]^{\frac{1}{3}}. \quad (32)$$

Substituting  $h(x)$  into equation (9) by its expression as a function of the Richardson number, the integration is straightforward and we obtain the longitudinal evolution of the Richardson number from the following implicit equation:

$$[4 - \text{Ri}(x)] \text{Ri}(x)^{1/3} = \frac{4}{3 \Lambda_0} x + [4 - \text{Ri}_0] \text{Ri}_0^{1/3}, \quad (33)$$

with

$$\Lambda_0 = \frac{h_0}{3 C_d \text{Ri}_0^{1/3}}. \quad (34)$$

### B. Constant entrainment coefficient

A constant entrainment coefficient (as in plumes and jets) can be associated with a strongly inertial gravity current for which the stable density gradient of the flow weakly affects the mixing, as in the case of a ceiling jet from an impinging plume. However, this model is no longer appropriate when the Richardson number increases and the effects of buoyancy appear (when  $\text{Ri}$  reaches 0.01 according to Christodoulou [34]).

From a mathematical point of view, returning to the equations (2), (3) and (4) with  $E = \alpha$ , a pure constant, the first derivatives of the primary variables and the local Richardson

number read:

$$\frac{d h}{d x} = \frac{\alpha}{2} \frac{3 + \kappa - \text{Ri}}{(1 - \text{Ri})}, \quad (35)$$

$$\frac{d u}{d x} = -\frac{\alpha}{2} \frac{u}{h} \frac{1 + \kappa + \text{Ri}}{(1 - \text{Ri})}, \quad (36)$$

$$\frac{d \eta}{d x} = -\alpha \frac{\eta}{h} \quad (37)$$

$$\frac{d \text{Ri}}{d x} = \frac{3 \alpha}{2} \frac{1}{h} \frac{\text{Ri}(1 + \kappa + \text{Ri})}{(1 - \text{Ri})}, \quad (38)$$

with  $\kappa = 1 + 2 C_d/\alpha$ .

In the same way as in Section III, we first express the primary variables as a function of the Richardson number:

$$\begin{aligned} \frac{u(x)}{u_0} &= \left[ \frac{\text{Ri}_0}{\text{Ri}(x)} \right]^{\frac{1}{3}}, \\ \frac{\eta(x)}{\eta_0} &= \left[ \frac{\text{Ri}(x) + 1 + \kappa}{\text{Ri}_0 + 1 + \kappa} \right]^{\frac{2(2+\kappa)}{3(1+\kappa)}} \left[ \frac{\text{Ri}_0}{\text{Ri}(x)} \right]^{\frac{2}{3(1+\kappa)}}, \\ \frac{h(x)}{h_0} &= \left[ \frac{\text{Ri}_0 + 1 + \kappa}{\text{Ri}(x) + 1 + \kappa} \right]^{\frac{2(2+\kappa)}{3(1+\kappa)}} \left[ \frac{\text{Ri}(x)}{\text{Ri}_0} \right]^{\frac{3+\kappa}{3(1+\kappa)}}, \end{aligned} \quad (39)$$

and by substituting  $h(x)$  in equation (38), we introduce the universal function  $F$ :

$$F(X) = \int_0^X \frac{(1 - \zeta) \zeta^{\frac{-2\kappa}{3(1+\kappa)}}}{(\zeta + 1 + \kappa)^{\frac{7+5\kappa}{3(1+\kappa)}}} d \zeta, \quad (40)$$

which allows the longitudinal evolution of the Richardson number to be determined from the following relation:

$$F[\text{Ri}(x)] = \frac{x}{\Lambda_0} + F(\text{Ri}_0), \quad (41)$$

where  $\Lambda_0$  is a constant equal to:

$$\Lambda_0 = \frac{2 h_0 (\text{Ri}_0 + 1 + \kappa)^{\frac{2(2+\kappa)}{3(1+\kappa)}}}{3 \alpha \text{Ri}_0^{\frac{(3+\kappa)}{3(1+\kappa)}}}. \quad (42)$$

### C. Entrainment law in the Parker-style

In their paper on turbidity currents, Parker *et al.* [29] proposed to express the entrainment coefficient as :

$$E = \frac{\alpha}{\beta + \text{Ri}}, \quad (43)$$

with  $\alpha$  and  $\beta$  two constants. This model is justified by the fact that it allows the entrainment coefficient to tend towards a universal asymptotic value for low Richardson numbers.

Mathematically, with this entrainment law, after a few algebra, the conservation equations allow the primary variables  $u(x)$ ,  $\eta(x)$  and  $h(x)$  to be written as a function of the local Richardson number  $\text{Ri}(x)$  as:

$$\begin{aligned}\frac{u(x)}{u_0} &= \left[ \frac{\text{Ri}_0}{\text{Ri}(x)} \right]^{\frac{1}{3}}, \\ \frac{\eta(x)}{\eta_0} &= \left[ \frac{\sigma + \kappa \text{Ri}(x)}{\sigma + \kappa \text{Ri}_0} \right]^{\frac{2(\kappa+\sigma)}{3\kappa\sigma}} \left[ \frac{\text{Ri}_0}{\text{Ri}(x)} \right]^{\frac{2}{3\sigma}}, \\ \frac{h(x)}{h_0} &= \left[ \frac{\sigma + \kappa \text{Ri}_0}{\sigma + \kappa \text{Ri}(x)} \right]^{\frac{2(\sigma+\kappa)}{3\kappa\sigma}} \left[ \frac{\text{Ri}_0}{\text{Ri}(x)} \right]^{\frac{2+\sigma}{3\sigma}},\end{aligned}\tag{44}$$

with  $\kappa = 1 + 2C_d/\alpha$  and  $\sigma = 2(1 - \beta + \beta\kappa)$ .

We then introduce the universal function  $F$ :

$$F(X) = \int_0^X \frac{(1 - \zeta)\zeta^{\frac{2-2\sigma}{3\sigma}}(\beta + \zeta)}{(\sigma + \kappa\zeta)^{\frac{\kappa(2+3\sigma)+2\sigma}{3\sigma\kappa}}} d\zeta,\tag{45}$$

which allows us to calculate the evolution of the Richardson number with the relation:

$$F[\text{Ri}(x)] = \frac{x}{\Lambda_0} + F(\text{Ri}_0),\tag{46}$$

where  $\Lambda_0$  is a constant defined as:

$$\Lambda_0 = \frac{2 h_0 (\sigma + \kappa \text{Ri}_0)^{\frac{2(\kappa+\sigma)}{3\sigma\kappa}}}{3 \alpha \text{Ri}_0^{\frac{2+\sigma}{3\sigma}}}.\tag{47}$$

For the sake of brevity, we do not discuss here the implementation of the results introduced in the section IV when a jump appears. However, it is entirely possible to combine these results with the methodology presented in section III B to adapt the entrainment law before or after the jump. A new function  $G$  is then obtained in order to determine the magnitude and the location of the jump.

## V. CONCLUSIONS AND DISCUSSIONS

This paper reports an analytical method to solve the equations of Ellison and Turner [21] describing the longitudinal evolution of a steady Boussinesq miscible gravity current. The



buoyant fluid that forms the current is continuously injected from a plane nozzle along a horizontal rigid boundary of finite length.

First, expressions of the primary variables of the current (velocity  $u$ , height  $h$  and density deficit  $\eta$ ) as explicit functions of the Richardson number  $\text{Ri}(x)$  are established. Then, the longitudinal evolution of the Richardson number is obtained from a universal function  $F$  (that depends only on the entrainment and drag coefficients). This universal function is given by an indefinite integral which can be easily calculated or, alternatively, tabulated. Here, we have illustrated the method by means of a graphical representation which, for prescribed injection conditions, allows the Richardson number value to be determined readily at any abscissa  $x$  along the horizontal boundary.

The ET59 equations presenting a mathematical singularity when  $\text{Ri} = 1$ , the developed method applies therefore easily as long as the flow remains supercritical ( $\text{Ri} < 1$ ) on the whole domain. Note that from the knowledge of the injection conditions and the length of the domain, it is possible to determine theoretically thanks to the universal function  $F$  whether the flow remains supercritical or should transition to a subcritical state ( $\text{Ri} > 1$ ).

If there is a transition (i.e. if the Richardson number reaches unity before the exit of the domain), the two regimes (supercritical and subcritical) can be matched by introducing in the ET59 equations a mathematical discontinuity similar to a hydraulic jump. For the sake of simplicity, we have modelled this jump using the Bélanger relation, as that was previously done by Haddad *et al.* [26]. In practice, the location of the jump is obtained from the injection conditions and a second universal function  $G$ . The evolution of the primary variables is then obtained for each zone (upstream and downstream of the jump) from the first function  $F$ .

The approach presented in this article has led to analytical solutions to the ET59 equations for different entrainment laws:  $E = 0$ ,  $E \propto \text{Ri}^0$ ,  $E \propto \text{Ri}^{-1}$  and  $E \propto (\beta + \text{Ri})^{-1}$ . It can also be applied to other laws proposed in the literature, such as those of ET59, Hebbert, Patterson, Loh, and Imberger [41], Dallimore *et al.* [30], Wells *et al.* [31], Johnson and Hogg [19] and van Reeuwijk *et al.* [42] for example.

Finally, in this article, we have studied theoretically a gravity current in a steady state, which is obviously the consequence of a transient development. In some configurations, we have had to introduce a mathematical discontinuity in the theoretical model (similar to a hydraulic jump). This raises the question of the existence (or not) and development of this

discontinuity in the transient phase. Ungarish [38] has recently addressed this issue using the shallow water equations. He has provided interesting results, but no definitive answers, making this a still-open question.

- 
- [1] P. C. Manins and B. L. Sawford, A model of katabatic winds, *J. Atmos. Sci.* **36**, 619 (1979).
- [2] C. Cenedese and C. Adduce, A new parameterization for entrainment in overflows, *J. Phys. Oceanogr.* **40**, 1835 (2010).
- [3] E. Meiburg and B. Kneller, Turbidity currents and their deposits, *Annu. Rev. Fluid Mech.* **42**, 135 (2010).
- [4] D. P. Hoult, Oil spreading on the sea, *Annu. Rev. Fluid Mech.* **4**, 341 (1972).
- [5] R. L. Alpert, Turbulent ceiling-jet induced by large-scale fires, *Combust. Sci. Technol.* **11**, 197 (1975).
- [6] T. Von Kármán, The engineer grapples with nonlinear problems, *Bull. Am. Math. Soc.* **46**, 615 (1940).
- [7] T. B. Benjamin, Gravity currents and related phenomena, *J. Fluid Mech.* **31**, 209 (1968).
- [8] H. E. Huppert and J. E. Simpson, The slumping of gravity currents, *J. Fluid Mech.* **99**, 785 (1980).
- [9] J. W. Rottman and J. E. Simpson, Gravity currents produced by instantaneous releases of a heavy fluid in a rectangular channel, *J. Fluid Mech.* **135**, 95 (1983).
- [10] R. J. Lowe, J. W. Rottman, and P. F. Linden, The non-Boussinesq lock-exchange problem. Part 1. Theory and experiments, *J. Fluid Mech* **537**, 101 (2005).
- [11] V. K. Birman, B. A. Battandier, E. Meiburg, and P. F. Linden, Lock-exchange flows in sloping channels, *J. Fluid Mech.* **577**, 53 (2007).
- [12] T. Bonometti, M. Ungarish, and S. Balachandar, A numerical investigation of high-reynolds-number constant-volume non-boussinesq density currents in deep ambient, *Journal of fluid mechanics* **673**, 574 (2011).
- [13] J. O. Shin, S. B. Dalziel, and P. F. Linden, Gravity currents produced by lock exchange, *J. Fluid Mech.* **521**, 1 (2004).
- [14] S. Longo, M. Ungarish, V. Di Federico, L. Chiapponi, and F. Addona, Gravity currents produced by constant and time varying inflow in a circular cross-section channel: experiments and theory, *Advances in water resources* **90**, 10 (2016).
- [15] D. Sher and A. W. Woods, Mixing in continuous gravity currents, *J. Fluid Mech.* **818**, R4 (2017).

- [16] A. Martin, M. Negretti, M. Ungarish, and T. Zemach, Propagation of a continuously supplied gravity current head down bottom slopes, *Physical Review Fluids* **5**, 054801 (2020).
- [17] M. Shringarpure, H. Lee, M. Ungarish, and S. Balachandar, Front conditions of high-re gravity currents produced by constant and time-dependent influx: an analytical and numerical study, *European Journal of Mechanics-B/Fluids* **41**, 109 (2013).
- [18] A. J. Hogg, M. M. Nasr-Azadani, M. Ungarish, and E. Meiburg, Sustained gravity currents in a channel, *Journal of Fluid Mechanics* **798**, 853 (2016).
- [19] C. G. Johnson and A. J. Hogg, Entraining gravity currents, *J. Fluid Mech.* **731**, 477 (2013).
- [20] M. Ungarish, Benjamin’s gravity current into an ambient fluid with an open surface, *J. Fluid Mech.* **825**, R1 (2017).
- [21] T. H. Ellison and J. S. Turner, Turbulent entrainment in stratified flows, *J. Fluid Mech.* **6**, 423–448 (1959).
- [22] B. R. Morton, G. I. Taylor, and J. S. Turner, Turbulent gravitational convection from maintained and instantaneous sources, *Proc. R. Soc. Lond.* **234**, 1 (1956).
- [23] D. L. Wilkinson and I. R. Wood, A rapidly varied flow phenomenon in a two-layer flow, *J. Fluid Mech.* **47**, 241 (1971).
- [24] Q. Guo, Y. Z. Li, H. Ingason, Z. Yan, and H. Zhu, Theoretical studies on buoyancy-driven ceiling jets of tunnel fires with natural ventilation, *Fire Saf. J.* **119**, 103228 (2021).
- [25] M. Dhar, G. Das, and P. K. Das, Planar hydraulic jumps in thin film flow, *J. Fluid Mech.* **884**, A11 (2020).
- [26] S. Haddad, S. Vaux, K. Varrall, and O. Vauquelin, Theoretical model of continuous inertial gravity currents including a jump condition, *Phys. Rev. Fluids* **7**, 084802 (2022).
- [27] G. Michaux and O. Vauquelin, Solutions for turbulent buoyant plumes rising from circular sources, *Phys. Fluids* **20**, 066601 (2008).
- [28] K. Lofquist, Flow and stress near an interface between stratified liquids, *Phys. Fluids* **3**, 158 (1960).
- [29] G. Parker, Y. Fukushima, and H. M. Pantin, Self-accelerating turbidity currents., *J. Fluid Mech.* **171** (1986).
- [30] C. J. Dallimore, J. Imberger, and T. Ishikawa, Entrainment and turbulence in saline underflow in lake ogawara, *Journal of Hydraulic Engineering* **127**, 937 (2001).
- [31] M. Wells, C. Cenedese, and C. P. Caulfield, The Relationship between flux coefficient and

- entrainment ratio in density currents, *J. Phys. Oceanogr.* **40**, 2713 (2010).
- [32] H. J. Fernando, Turbulent mixing in stratified fluids, *Annu. Rev. Fluid Mech.* **23**, 455 (1991).
- [33] M. R. Chowdhury and F. Y. Testik, A review of gravity currents formed by submerged single-port discharges in inland and coastal waters, *Environ. Fluid Mech.* **14**, 265 (2014).
- [34] G. C. Christodoulou, Interfacial mixing in stratified flows., *J. Hydraul. Res.* **24**, 77 (1986).
- [35] H. I. Nogueira, C. Adduce, E. Alves, and M. J. Franca, Dynamics of the head of gravity currents, *Environ. Fluid Mech.* **14**, 519 (2014).
- [36] A. J. Hogg and A. W. Woods, The transition from inertia-to bottom-drag-dominated motion of turbulent gravity currents, *J. Fluid Mech.* **449**, 201 (2001).
- [37] P. G. Baines, Mixing regimes for the flow of dense fluid down slopes into stratified environments, *J. Fluid Mech.* **538**, 245 (2005).
- [38] M. Ungarish, Strongly supercritical non-boussinesq sustained gravity currents: Time-dependent and steady-state approximate solutions, *Phys. Rev. Fluids* **8**, 053801 (2023).
- [39] S. Kostic and G. Parker, Conditions under which a supercritical turbidity current traverses an abrupt transition to vanishing bed slope without a hydraulic jump, *J. Fluid Mech.* **586**, 119 (2007).
- [40] F. M. Henderson, *Open channel flow*, Macmillan Series in Civil Engineering Series (Macmillan, New York, 1966).
- [41] B. Hebbert, J. Patterson, I. Loh, and J. Imberger, Collie river underflow into the wellington reservoir, *Journal of the Hydraulics Division* **105**, 533 (1979).
- [42] M. van Reeuwijk, M. Holzner, and C. Caulfield, Mixing and entrainment are suppressed in inclined gravity currents, *J. Fluid Mech.* **873**, 786 (2019).